Triangles with largest integer semiperimeter.

Problem with a solution proposed by Arkady Alt , San Jose , California, USA.

For given integer numbers $a_1, a_2, a_3 \geq 2$ find the largest value of integer semiperimeter of a triangle with integer sidelengths t_1, t_2, t_3, satisfying inequalities $t_i \leq a_i, i = 1,2,3$.

Solution.

Let $s := \frac{t_1 + t_2 + t_3}{2}$. Since $t_i < s, i = 1,2,3$ (triangle inequalities) then our problem is:

Find $\max s$ for which there are positive integer numbers t_1, t_2, t_3 satisfying $t_i \leq \min\{a_i, s-1\}, i = 1,2,3$ $t_1 + t_2 + t_3 = 2s$.

First note that $s \geq 3, t_i \geq 2, i = 1,2,3$. Indeed, since $t_i \leq s - 1$then $1 \leq s - t_i, , i = 1,2,3$ and, therefore, $t_1 = 2s - t_2 - t_3 = (s - t_2) + (s - t_3) \geq 2$. Cyclic we obtain $t_2, t_3 \geq 2$. Hence, $2s \geq 6 \Leftrightarrow s \geq 3$.

Since $t_3 = 2s - t_1 - t_2,\ \ 2 \leq t_3 \leq \min\{a_3, s-1\}$ then $1 \leq 2s - t_1 - t_2 \leq \min\{a_3, s-1\} \Leftrightarrow$ $\max\{2s - t_1 - a_3, s + 1 - t_1\} \leq t_2 \leq 2s - 1 - t_1$ and, therefore, for t_2 we obtain inequality

(**1**) $\quad \max\{2s - t_1 - a_3, s + 1 - t_1, 2\} \leq t_2 \leq \min\{2s - 1 - t_1, a_2, s - 1\}$

with conditions of solvability :

$$(\mathbf{2}) \quad \begin{cases} 2s - t_1 - a_3 \leq s - 1 \\ 2s - t_1 - a_3 \leq a_2 \\ s + 1 - t_1 \leq a_2 \\ 2 \leq 2s - 1 - t_1 \end{cases} \Leftrightarrow \begin{cases} s + 1 - a_3 \leq t_1 \\ 2s - a_2 - a_3 \leq t_1 \\ s + 1 - a_2 \leq t_1 \\ t_1 \leq 2s - 3 \end{cases}.$$

Since $s - 1 \leq 2s - 3$ then (**2**) together with $2 \leq t_1 \leq \min\{a_1, s-1\}$ it gives us bounds for t_1 :

(**3**) $\max\{s + 1 - a_3, 2s - a_2 - a_3, s + 1 - a_2, 2\} \leq t_1 \leq \min\{a_1, s - 1\}$.

Since $2 \leq a_i, i = 2,3$ then $s + 1 - a_2 \leq s - 1, s + 1 - a_3 \leq s - 1$ and solvability condition for (**3**) becomes

$s + 1 - a_3 \leq a_1 \Leftrightarrow s \leq a_1 + a_3 - 1, 2s - a_2 - a_3 \leq a_1 \Leftrightarrow s \leq \left\lfloor \frac{a_1 + a_2 + a_3}{2} \right\rfloor$,

$s + 1 - a_2 \leq a_1 \Leftrightarrow s \leq a_1 + a_2 - 1, 2s - a_2 - a_3 \leq s - 1 \Leftrightarrow s \leq a_2 + a_3 - 1$.

Thus, $s^* = \min\left\{\left\lfloor \frac{a_1 + a_2 + a_3}{2} \right\rfloor, a_1 + a_2 - 1, a_2 + a_3 - 1, a_3 + a_1 - 1\right\}$ is the largest value of integer semiperimeter.

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